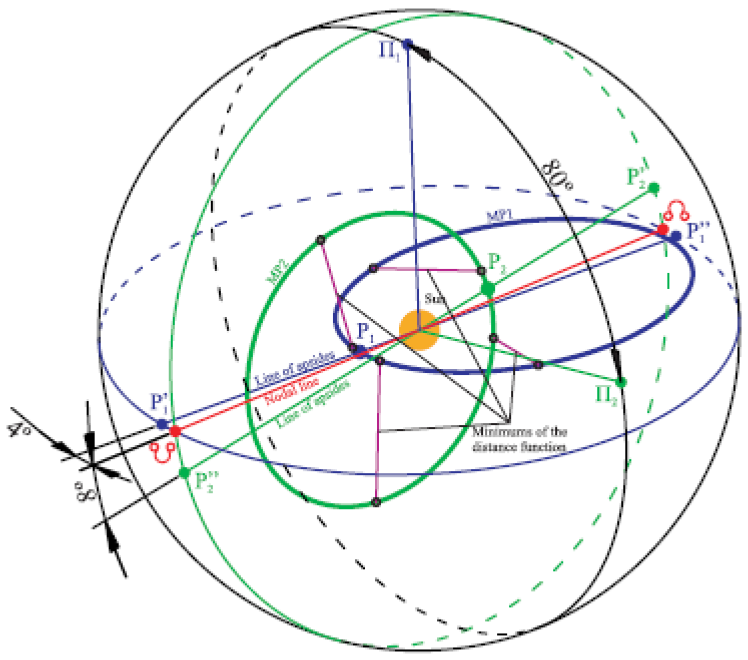

PROXIMITY CALCULATION AND CHANGING OF DISTANCE FUNCTION

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Abstract: The paper discusses the analytical expression for the number and type of proximities of asteroids and changing of distance function for some characteristic simulations pairs of elliptical orbits. Shows that the extreme values of critical points cannot be found in such groups.

1. Introduction

Proksimity problem dates back to the mid-nineteenth century, but it is still current whether is it about different approaches to calculate it or about the application of that account in order to determinate their size and number.

Many famous astronomers of the time dealing with this problem B. A. Gould, H. A. d'Arest, K. V. Littrow, E. Strömgrem, Gauss, Linsser, A. Galle, G. Fayet. Lazović (1974, 1976) gave some results concerning the duration of the proximities, as well as the changes of the mutual distances caused by the changes of the orbital elements. In the case of proximities for the orbits of asteroids Ceres, Palas, Juno, Vesta and others, Lazović and Kuzmanoski (1983) obtain a proximity of only 0.0000154 a.u. between (2) Palas and (1193) Afrika. Recently we met different approaches to the proximity problem. Kholshevnikov and Vassiliev (1999) succeeded in simplifying the function of distance between two orbits by applying various substitutions. Gronchi (2002) determined the total, *i.e.*, the maximum possible number, of stationary points in the distance function and gave the dependence of their number on the shape of their orbits, which is shown in Table 1. These analyzes confirm the earlier hypotheses that there are maximum four minima, or proximities, in the distance function between two confocal elliptical orbits.

Table 1

Eccentricity of first orbit	Eccentricity of second orbit	Maximum possible number of stationary points
$e1 \neq 0$	$e2 \neq 0$	12
$e1 \neq 0$	$e2 = 0$	10
$e1 = 0$	$e2 \neq 0$	10
$e1 = 0$	$e2 = 0$	8

2. Analytical expression

There are several papers in the literature on the computation of the minimum distance (Sitarski 1968, Hoots 1984, Dybczynski *et al.* 1986).

By limiting the treatment to the cases of coplanar asteroids or, so called quasicoplanar pairs or asteroid groups, the most complete method for calculating the proximities was given by Lazović (1964). From all of the previous analyzes, it became clear that:

• There is always at least one proximity
• Two proximities can exist and this is, practically, the most frequent case
• The case with three proximities is possible, but it takes place much more rarely when the orbits of asteroids are studied
• Four proximities are also possible, but in reality it is very difficult to find them

Our procedure for finding all possible proximities, is based on an idea of Simovljević (1974). According to this idea one should use the fact that the two position vectors $\vec{r}_1$ and $\vec{r}_2$ are known at every point of the orbit which yields the relative position vector $\vec{\rho}$. Its

modulus at the point of the proximity determines the magnitude
of the proximity itself. Without introducing any approximations we establish the convergence of the relative position vector ρ provided that the limiting case
of its convergence is always at one of the possible proximities or maximums. In this way by calculating the smallest value of the relative position vector we can also calculate the proximity. We shall have as many proximities as there are such convergence during the revolution along the orbits. The equations that must be satisfied at every proximity are vector equations which define orthogonality condition between relative position vector $\rho = r_1 - r_2$ and tangents on both orbits.
The equations that must be satisfied at every proximity are vector equations which define orthogonality condition between relative position vector $\rho = r_1 - r_2$
and tangents on both orbits.

$$(\vec{r}_2 - \vec{r}_1) \cdot \frac{\mathrm{d}\vec{r}_2}{\mathrm{d}t} = 0,$$
$$(\vec{r}_1 - \vec{r}_2) \cdot \frac{\mathrm{d}\vec{r}_1}{\mathrm{d}t} = 0.$$

(1)

The position vectors of the asteroids can be expressed in the terms of the eccentric anomalies *E*1 and *E*2 , *i.e.*,

$$\begin{aligned}\vec{r}_1 &= a_1(\cos E_1 - e_1)\vec{P}_1 + b_1 \sin E_1 \vec{Q}_1,\\ \vec{r}_2 &= a_2(\cos E_2 - e_2)\vec{P}_2 + b_2 \sin E_2 \vec{Q}_2\end{aligned}$$

(2)

Using relation $\frac{dr}{dt} = \frac{dr}{dE}\frac{dE}{dt}$, Eqs. (1) can be transformed to

$$(\vec{r}_1 - \vec{r}_2) \cdot \frac{\mathrm{d}\vec{r}_1}{\mathrm{d}E_1} = 0,$$
$$(\vec{r}_2 - \vec{r}_1) \cdot \frac{\mathrm{d}\vec{r}_2}{\mathrm{d}E_2} = 0.$$

(3)

After some substitutions (Lazović 1993) and grouping terms, one obtains the following expression:

$$A \sin E_2 + B \cos E_2 + C \sin E_2 \cos E_2 = 0$$

(4)

A, *B* and *C* are constants values which depend from orbits elements and one eccentric anomaly. If Eq. (4) is divided by the coefficient *B* (*B* ≠ 0), one obtains:

$$M \sin E_2 + \cos E_2 + N \sin E_2 \cos E_2 = 0$$

(5)

where *M* and *N* are the ratios *A/B* and *C/B*, respectively.
After using couple trigonometric equations and rationalization one obtains a fourth-order equation

$$a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t^1 + a_0 = 0,$$

(6)

where we have,

$$a_0 = -1,\ a_1 = -2(M+N),\ a_2 = 0,$$
$$a_3 = -2(M-N),\ a_4 = 1.$$

An important property of this equation is that it, with regard to its form and the problem which describes, must always have 2 or 4 real roots. Only one of the four possible roots is of interest to us. It is defined as the
smallest value among all possible modules of the differences of vectors r_1 and r_2.

3. Simulation procedure

As already seen, the present procedure is reduced to the solving of a characteristic equation of the fourth order, which is always solvable. The procedure is easily applicable to any pair of elliptic orbits with no regards to their relative position or, more precisely, to the orbital elements. As its final result this method yields the values of *E*1 , *E*2 and ρ for each proximity and gives the total number of proximities. Generally, the aim is efficiency of finding proximities and determination of their positions and magnitudes as accurately as possible with a minimum expenditure of time and computing resources and applying it to as many simulation pairs. In the
calculation we have used a program written in MATLAB 7.0. By moving along both orbits and by successive solving Eq. (6) we obtained two functions which
represent minimal distance between two orbits as a function of eccentric anomalies:

$\rho_{1\min} = \rho_{1\min}(E_1),\ E_1 \in [0,2\pi],$
$\rho_{2\min} = \rho_{2\min}(E_2),\ E_2 \in [0,2\pi].$

For these functions the local minima and maxima which represent the extremes of the distance function are determined then. It is necessary to combine these two functions because the solution with the smallest module of relative position vector ρ of the first equation from the Eqs. (3) does not have to be also the solution with the smallest value of module of ρ of the second equation from the Eqs. (3) at the same point. This situation, which is extremely rare, is shown in Fig. 1. This is why we have written, in Section 4 and distance function ρ as function of the two (alternative) arguments. By analyzing these two functions we can determine number and locations off all critical points and also the magnitudes of the proximities and the maxima of the distance function.

4. Applying the Method to a Selected Set of Simulations

By application of the procedure presented above we have examined the proximities and maximums of the distance function between 1136441 simulation pairs (for each type in three groups of two elliptical orbits with common focus). The values of orbit elements and results are shown in Table 2 where, for the instance, we placed the Gronchi's (2002) analytically calculated upper limit of stationary points at $n \leq 16$ but our numerical algorithm for searching of orbital configurations maximizes the number of stationary points at $n = 12$.

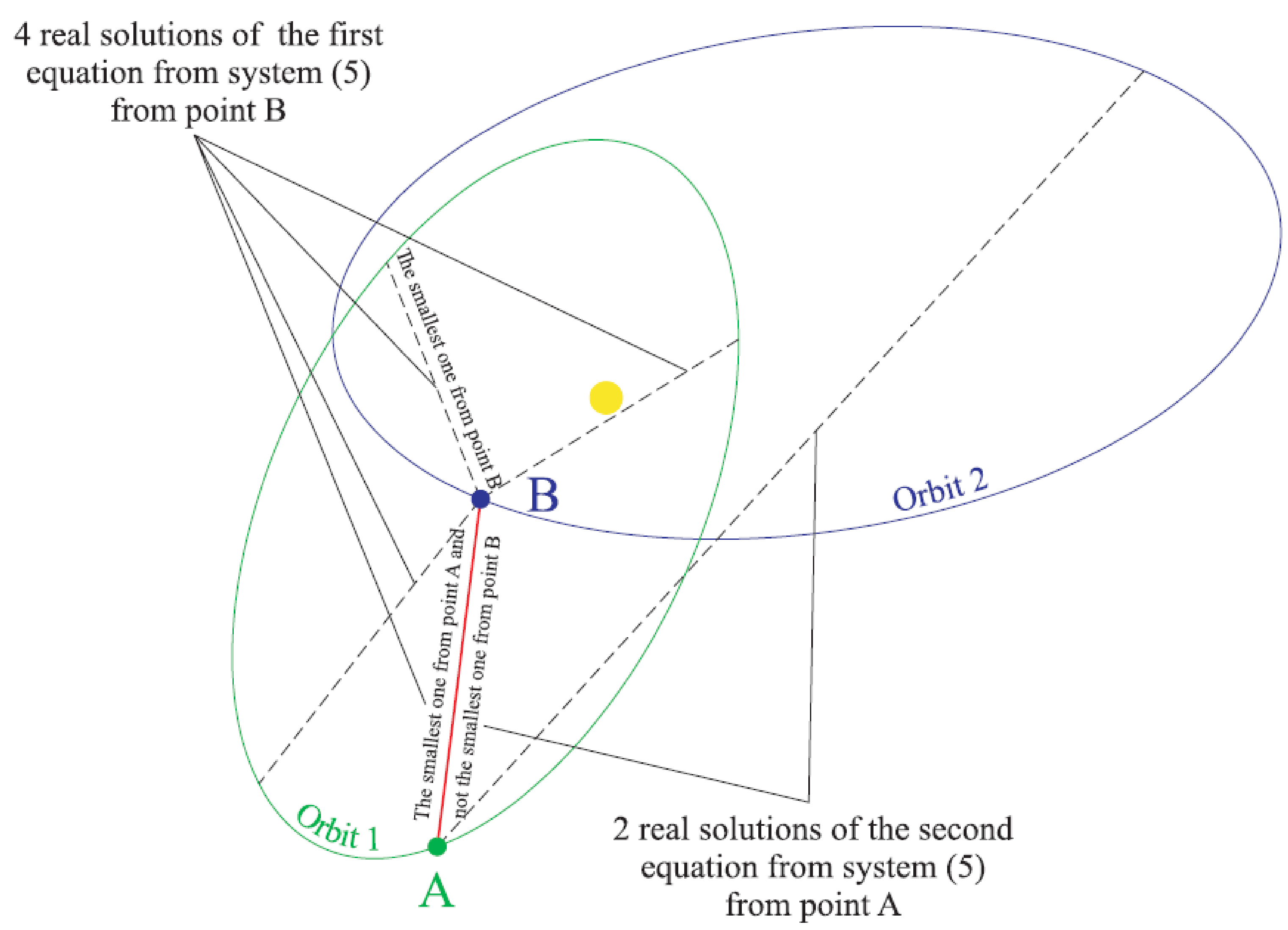
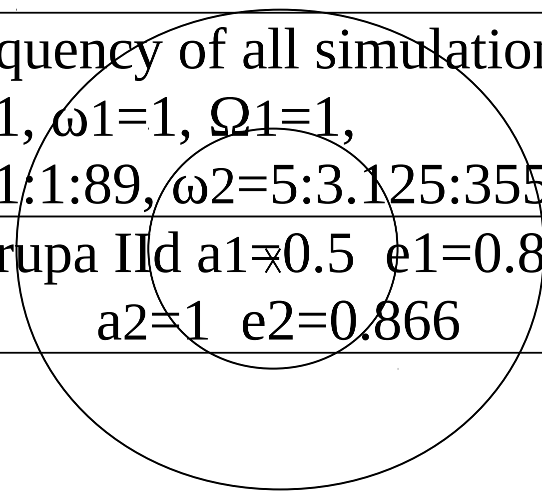
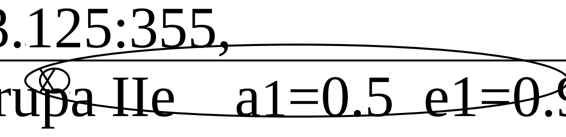
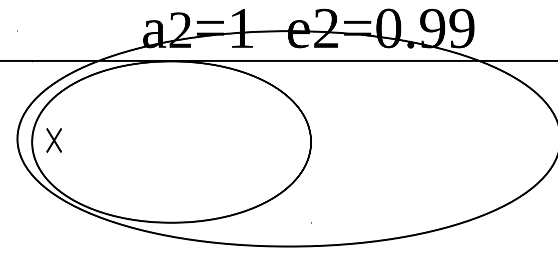
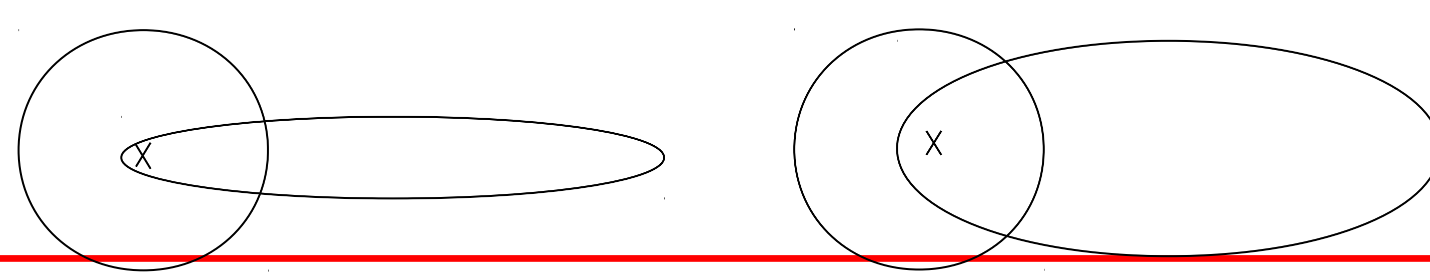


Fig. 1. Geometrical interpretation of the system of Eqs. (4).

Table 2

Min - Max	%	Min - Max	%	Frequency of all simulations with next orbit elements: i1=1, ω1=1, Ω1=1, i2=1:1:89, ω2=5:3.125:355, Ω2=5:3.125:355,		Min - Max	%	Min - Max	%
				Grupa Ia a1=1 e1=0.01 a1=2 e2=0.01	Grupa Ib a1=1 e1=0.866 a2=1 e2=0.866				
1-1	-	3-1	-			1-1	11	3-1	-
1-2	-	3-2	-			1-2	-	3-2	2
1-3	-	3-3	-			1-3	-	3-3	-
1-4	-	3-4	-			1-4	-	3-4	-
2-1	13	4-1	-			2-1	53	4-1	-
2-2	87	4-2	-			2-2	30	4-2	1
2-3	-	4-3	-			2-3	3	4-3	-
2-4	-	4-4	-			2-4	-	4-4	-
				Grupa IIa a1=0.5 e1=0.01	Grupa IIb a1=0.01 e1=0.01				

				a2=1 e2=0.01	a2=1 e2=0.99				
1-1	7	3-1	-			1-1	-	3-1	13
1-2	-	3-2	-			1-2	-	3-2	33
1-3	-	3-3	-			1-3	-	3-3	-
1-4	-	3-4	-			1-4	-	3-4	-
2-1	8	4-1	-			2-1	51	4-1	2
2-2	85	4-2	-			2-2	-	4-2	1
2-3	-	4-3	-			2-3	-	4-3	-
2-4	-	4-4	-			2-4	-	4-4	-
Min - Max	%	Min - Max	%	Frequency of all simulations with next orbit elements: i1=1, ω1=1, Ω1=1, i2=1:1:89, ω2=5:3.125:355, Ω2=5:3.125:355,		Min - Max	%	Min - Max	%
				Grupa IId a1=0.5 e1=0.866 a2=1 e2=0.866 	Grupa IIe a1=0.5 e1=0.99 a2=1 e2=0.99 				
1-1	-	3-1	-			1-1	-	3-1	-
1-2	62	3-2	2			1-2	19	3-2	8
1-3	-	3-3	-			1-3	-	3-3	-
1-4	-	3-4	-			1-4	-	3-4	-
2-1	-	4-1	-			2-1	-	4-1	2
2-2	34	4-2	-			2-2	71	4-2	-
2-3	2	4-3	-			2-3	-	4-3	-
2-4	-	4-4	-			2-4	-	4-4	-
				Grupa IIId a1=0.5 e1=0.01 a2=1 e2=0.99	Grupa IIIe a1=0.5 e1=0.01 a2=1 e2=0.866				
1-1	-	3-1	-			1-1	-	3-1	19
1-2	-	3-2	86			1-2	-	3-2	37
1-3	-	3-3	-			1-3	-	3-3	-
1-4	-	3-4	-			1-4	-	3-4	-
2-1	14	4-1	-			2-1	44	4-1	-
2-2	-	4-2	-			2-2	-	4-2	-
2-3	-	4-3	-			2-3	-	4-3	-
2-4	-	4-4	-			2-4	-	4-4	-



5. Conclusions

We present a combined, analytical and numerical method of solving the problem of the minimum distance between two elliptical orbits with a common focus. Analyzing all the obtained results the following comments become possible:

- The most frequency simulation pairs are with 1min 1max and 2min 1max
- The pairs with one and two minimum distances (proximities) make more than 96 percent of the total number of the simulation pairs
- The pairs with 4 max are didn't find
- The pairs with 14 and 16 stationary points also didn't find (12 stationary points is the best upper bound solutions which has been explicitly given by Gronchi 2002)
- Requirement that the two elliptical orbits have or don't have intersecting points when they are in the same plane haven't influence on the number of stationary points of distance function.

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